SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2012
HIGHER SCHOOL CERTIFICATE
ASSESSMENT TASK #3

Mathematics

General Instructions

- Reading Time – 5 Minutes
- Working time – 2 hours
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators may be used.
- Each Section (A, B, and C) is to be returned in a separate bundle.
- All necessary working should be shown in every question, except multiple choice.

Total Marks – 75

- Attempt questions 1 – 6.
- All questions are not of equal value.
- Unless otherwise directed give your answers in simplest exact form.

Examiner: *A.M.Gainford*

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1} x^{n+1}, \; n \neq -1; \; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \; x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}, \; a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \; a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \; a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \; a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \; a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \; a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \; a > 0, \; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \; x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \; x > 0$

Section A (22 Marks)

START A NEW BOOKLET

Question 1. (7 marks)

Indicate which of the answers A, B, C, or D is the correct answer. Write the letter corresponding to the answer in your answer booklet. **Marks**

(a) Solve for y: $xy - d = m$:- **1**

A: $y = \frac{m-d}{x}$

B: $y = m + d - x$

C: $y = \frac{m+d}{x}$

D: $xy = m + d$

(b) Reduce the following expression to simplest form: **1**

$$\frac{12a^3c}{4ac} =$$

A: $8a^2$

B: $3a^2$

C: $3a^3$

D: $3a^3c$

(c) The derivative of $y = 6x^{-3}$ equals:- **1**

A: $-18x^{-4}$

B: $-18x^{-2}$

C: $-12x^{-3}$

D: $-3x^{-2}$

(d) Find a primitive of $\frac{1}{2x+3}$:- **1**

A: $\frac{2}{(2x+3)^2}$

B: $\ln(2x+3)$

C: $\frac{1}{2}\ln(2x+3)$

D: $\ln(2x)+3$

(e) If $(2,-3)$ is the midpoint of the interval joining $A(4,-2)$ and $B(x,y)$ then B has co-ordinates:- **1**

A: $(-4,0)$

B: $(0,0)$

C: $(0,-4)$

D: $(0,-8)$

(f) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos 2x\,dx =$ **1**

A: $\frac{1}{2}$

B: 0

C: 1

D: $-\frac{1}{2}$

(g) The derivative of $\ln\sqrt{1-x^2}$ is:- **1**

A: $\dfrac{1}{2\sqrt{1-x^2}}$

B: $\dfrac{-x}{1-x^2}$

C: $\dfrac{-x}{\sqrt{1-x^2}}$

D: $\dfrac{x}{\sqrt{x^2-1}}$

Question 2 (15 marks)

Marks

(a) Differentiate the following: **4**

(i) $y = 2\tan 3x$

(ii) $y = x\sin x$

(iii) $y = e^{\sin x}$

(iv) $y = \ln 4x$

(b) Find **4**

(i) $\int \left(1 - e^{x}\right)^{2} dx$

(ii) $\int_{\frac{1}{6}}^{\frac{1}{2}} \cos \pi x \, dx$

(c) Use the trapezoidal rule with four function values to find an approximation to **3**

$$\int_{-2}^{1} 2^{-x} \, dx$$

(Answer correct to two decimal places.)

(d) The area enclosed between $y = 2x - x^2$ and $y = x$ is rotated about the x-axis. **4**

(i) Find the points of intersection of the curves.

(ii) Find the volume of the solid generated.

Section B (26 Marks)
START A NEW BOOKLET

Question 3 (13 Marks)

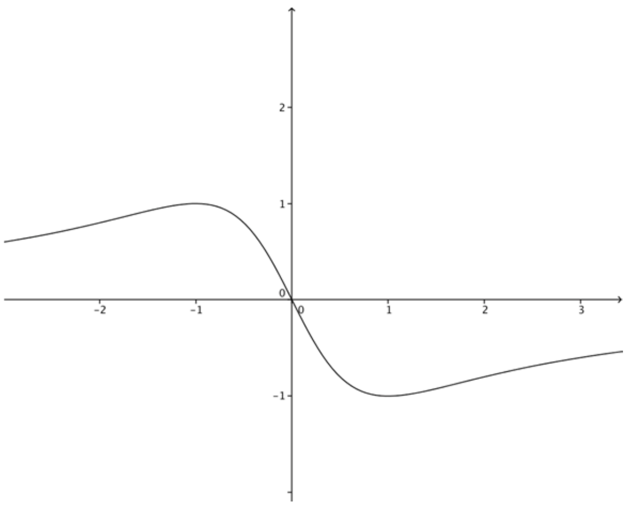
Marks

(a) Find the area of a segment of a circle of radius 4 cm and angle at the centre 1.5 radians, correct to two decimal places. **3**

(b) Simplify: $\dfrac{1}{\sec^2 x} + \dfrac{1}{\text{cosec}^2 x}$ **1**

(c) Sketch the graph of $y = 3\cos 2x$ in the domain $0 \le x \le 2\pi$. **2**

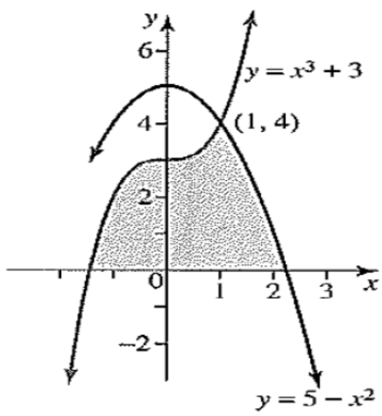
(d) The graph shows $f'(x)$, the derivative of $y = f(x)$. **4**



(i) Copy the graph to your answer booklet.

(ii) Sketch on the graph a possible graph of $y = f(x)$ given that $f(x) > 0$ for all x. On your graph of $y = f(x)$ mark any points of inflexion.

(e) **3**



Find the area bounded by the curves $y = x^3 + 3$, $y = 5 - x^2$ and the x-axis.

Question 4 (13 Marks)

(a) For a function $f(x)$ it is given that $f(0)=4$ and $f(1)=12$. **3**

Find, in simplest form, the value of:

$$\int_0^1 \frac{f'(x)}{f(x)}\,dx$$

(b) Given the function $y = 2 + 3x - x^3$: **6**

(i) Find the co-ordinates of the stationary points, and determine their nature.

(ii) Find the co-ordinates of any points of inflexion.

(iii) Sketch the curve in the domain $-3 \le x \le 3$.

(c) Find the equation of the normal to the curve $y = e^{-x}$ at the point where $x = 1$. **2**

(d) The Fundamental Theorem of the Integral Calculus is: **2**

$$\int_a^b f(x)dx = F(b) - F(a)$$

where $F(x)$ is a primitive of $f(x)$.

Use this theorem to prove that:

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx$$

Section C (27 Marks)

START A NEW BOOKLET

Question 5 (13 Marks)

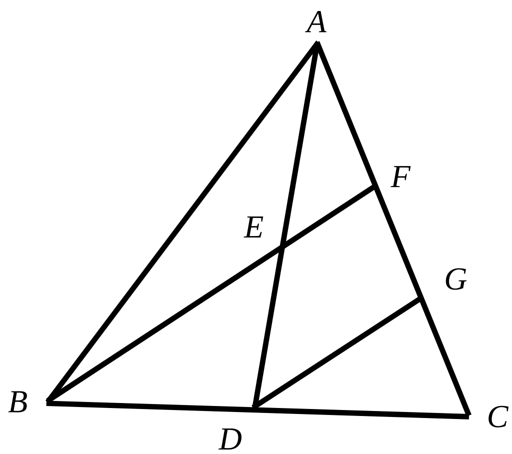
(a) A sphere of volume $V = \frac{4}{3}\pi r^3$ and surface area $A = 4\pi r^2$ is expanding at a constant rate. Find the ratio of the rate of change of volume with respect to *r* to the rate of change of surface area with respect to *r*. **2**

(b) For the triangle with vertices $A(-3,0)$, $B(2,4)$, and $C(6,-1)$: **6**

(i) Show that the triangle is isosceles.

(ii) Find the area of the triangle.

(c) **5**



In the diagram, *D* and *E* are the midpoints of *BC* and *AD* respectively, and $DG \parallel BF$.

(i) Copy the diagram to your answer booklet.

(ii) Prove that $AF = FG = GC$.

Question 6 (14 Marks)

(a) Simplify $\ln\left(\ln\sqrt{e^4}\right)$. **1**

(b) A particle is moving in a straight line with acceleration given by $a = 12t$ m/s^2. **4**
Initially the particle is at the origin with velocity -8 m/s .

(i) Find the equation for its position at time t.
.

(ii) State when the particle is next at the origin.

(c) (i) Find $\dfrac{d}{dx}\left(\dfrac{3}{x^3+1}\right)$. **4**

(ii) The region of the number plane under the curve $y = \dfrac{3x}{x^3+1}$, above the x-axis, and between $x = 0$ and $x = 1$ is rotated about the x-axis. With the aid of part (i), find then volume of the solid formed.

(d) The volume of a closed cylindrical container is 800 m^3. The container is to be made of aluminium sheeting costing $78 per square metre. **5**

(i) Show that, given the radius of the base is r, the area of aluminium required is

$$A = 2\pi r^2 + \frac{1600}{r}$$

(ii) Find the minimum cost (to the nearest dollar) of the aluminium for making the container.

This is the end of the paper.

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Q1.

a) C b) B c) A

d) C e) C f) C

g) B

Q2

a) i) $y = 2\tan 3x$

$y' = 6\sec^2 3x$

ii) $y = x\sin x$

$y' = \sin x(1) + x(\cos x)$

iii) $y = e^{\sin x}$

$y' = \cos x \, e^{\sin x}$

iv) $y = \ln 4x$

$y' = \frac{4}{4x} = \frac{1}{x}$

b) i) $\int (1-e^{x})^2 \, dx$

$= \int (1 - 2e^{x} + e^{2x}) \, dx$

$= x - 2e^{x} + \frac{e^{2x}}{2} + C$

b) ii) $\int_{\frac{1}{6}}^{\frac{1}{2}} \cos \pi x \, dx$

$= \left[\frac{1}{\pi}\sin \pi x\right]_{\frac{1}{6}}^{\frac{1}{2}}$

$= \frac{1}{\pi}\left(\sin\frac{\pi}{2} - \sin\frac{\pi}{6}\right)$

$= \frac{1}{\pi}\left(1 - \frac{1}{2}\right)$

$= \frac{1}{2\pi}$ //

c) $h = \frac{b-a}{n}$ $n = 3$

$= \frac{1 - {}^{-}2}{3} = {}^{+}1$

$\int_{-2}^{1} 2^{-x} \, dx = \frac{h}{2}\left(y_0 + y_3 + 2(y_1 + y_2)\right)$

	y_0	y_1	y_2	y_3
x	-2	-1	0	1
y	4	2	1	$\frac{1}{2}$

$= \frac{1}{2}\left(4 + \frac{1}{2} + 2(2+1)\right)$

$= \frac{1}{2}\left(10\frac{1}{2}\right)$

$\doteqdot 5.25$

d) $V = \pi\int_0^1 (2x - x^2)^2 \, dx - \pi\int_0^1 x^2 \, dx$

$= \pi\int (4x^2 - 4x^3 + x^4) \, dx - \pi\int x^2 \, dx$

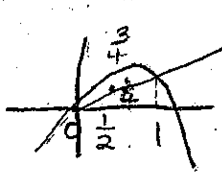
$= \pi\left[\frac{4}{3}x^3 - \frac{4}{4}x^4 + \frac{x^5}{5}\right] - \pi\left(\frac{x^3}{3}\right)$

$= \pi\left[\frac{4}{3} - 1 + \frac{1}{5}\right] - \pi\left[\frac{1}{3}\right] - 0$

$= \frac{\pi}{5}$ units^3

D) Points of intersection

$2x - x^2 = x$



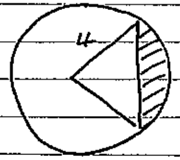
$0 = x^2 - x$

$0 = x^2 - x$

$0 = x(x-1)$

$x = 0$ $x = 1$

3(a)



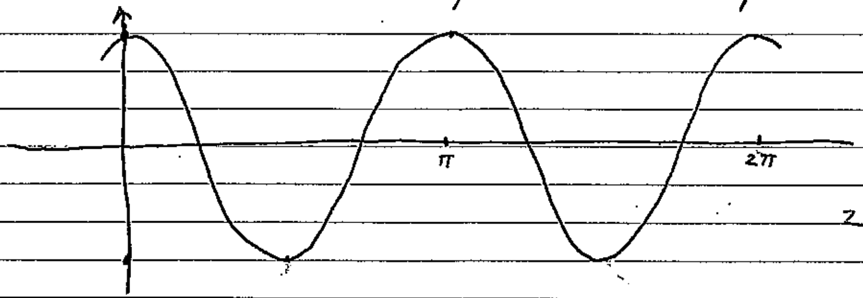
Area of Minor Segment $= \frac{1}{2}(r^2(\theta - \sin\theta))$
$= 8(1.5 - \sin 1.5)$
$= 4.02$ 3

(Major Segment = 46.25)

(b) $\frac{1}{\sec^2 x} + \frac{1}{\csc^2 x} = \cos^2 x + \sin^2 x$

$= 1$ 1

(c) $y = 3\cos 2x$ has amplitude 3 and period π.



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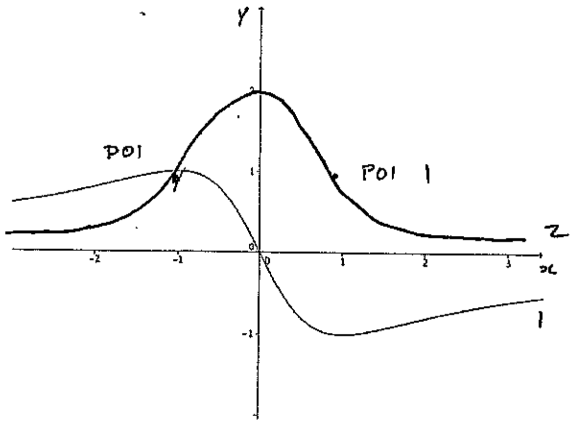
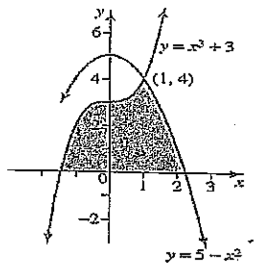
(d) (i)
(ii)



4 2 1

Question 3 (e)



$y = x^3 + 3$ cuts x-axis at $x = \sqrt[3]{3}$

$y = 5 - x^2$ cuts x-axis at $x = \sqrt{5}$

Area $= \int_{-\sqrt[3]{3}}^{1} x^3 + 3 \, dx + \int_{1}^{\sqrt{5}} 5 - x^2$ 3

$= \left[\frac{1}{4}x^4 + 3x\right]_{-\sqrt[3]{3}}^{1} + \left[5x - \frac{x^3}{3}\right]_{1}^{\sqrt{5}}$

$= 6.50 + 2.79 = 9.29$

4 (a) $\int_0^1 \frac{f'(x)}{f(x)} \, dx = \left[\ln f(x)\right]_0^1$

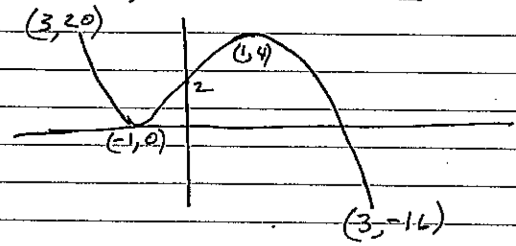
$= \ln f(1) - \ln f(0)$ 3
$= \ln 12 - \ln 4$
$= \ln 3$

(b) $y = 2 + 3x - x^3$
$\frac{dy}{dx} = 3 - 3x^2$
$\frac{d^2y}{dx^2} = -6x$ 1

Stat Pts. $\frac{dy}{dx} = 0$
$\therefore 3 - 3x^2 = 0$
$x = \pm 1$ 2

at $x = 1$; $\frac{d^2y}{dx^2}$ is $-ve$ $\therefore$ (1, 4) is Max. Turn Pt
at $x = -1$; $\frac{d^2y}{dx^2}$ is $+ve$ $\therefore$ (−1, 0) is Min. Turn Pt

P.O.I at $\frac{d^2y}{dx^2} = 0$ $\therefore$ at $x = 0$
Before $x = 0$ $\frac{d^2y}{dx^2}$ is $+ve$ and after $x = 0$ $\frac{d^2y}{dx^2}$ is $-ve$ 1
$\therefore$ Concavity changes
(0, 2) is P.O.I



2

(c) $y = e^{-x}$
$\frac{dy}{dx} = -e^{-x}$

at $x = 1$, $y = \frac{1}{e}$ and gradient of normal is $\frac{-1}{-e^{-1}} = e$

$\therefore$ Eqn is $y - \frac{1}{e} = e(x - 1)$
$ey - 1 = e^2(x - 1)$
$ey - e^2x - 1 + e^2 = 0$

4 (d) Prove $\int_0^a f(x)\,.dx = \int_0^a f(a-x)\,.dx$

LHS $= F(a) - F(0)$

RHS $= \left[-F(a-x)\right]_0^a$

$= -F(0) + F(a)$

$= F(a) - F(0)$

$=$ LHS

Q.E.D.

Question 5

a) $V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2$

$A = 4\pi r^2$

$\frac{dA}{dr} = 8\pi r$

$\frac{dV}{dr} : \frac{dA}{dr}$

$= 4\pi r^2 : 8\pi r$

$= r : 2$

b) i) $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$AB = \sqrt{(2-(-3))^2 + (4-0)^2}$
$= \sqrt{41}$

$AC = \sqrt{(6-(-3))^2 + (-1-0)^2}$
$= \sqrt{82}$

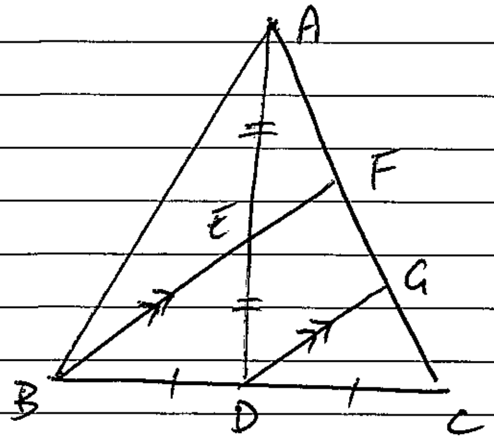
$BC = \sqrt{(2-6)^2 + (4-(-1))^2}$
$= \sqrt{41}$

since $AB = BC$
$\therefore \triangle ABC$ is isosceles

ii) since pythagoras' theorem holds
ie $AB^2 + BC^2 = AC^2$
$\triangle ABC$ is right angled
$\therefore A = \frac{1}{2} \times \sqrt{41} \times \sqrt{41}$

$= \frac{41}{2}$ units^2

c) i)



ii) $\frac{FG}{GC} = \frac{BD}{DC}$ (ratio of intercepts, $DG \parallel BF$)

$BD = DC$ (D is midpoint of BC)

$\therefore \frac{FG}{GC} = 1$

$FG = GC$

$\frac{AF}{FG} = \frac{AE}{ED}$ (ratio of intercepts, $DG \parallel BF$)

$AE = ED$ (E is midpoint of AD)

$\therefore \frac{AF}{FG} = 1$

$AF = FG$

And so $AF = FG = GC$

Question 6

a) $\ln(\ln\sqrt{e^4})$

$= \ln(\ln(e^4)^{\frac{1}{2}})$

$= \ln(\ln e^2)$

$= \ln 2$

b) i) $a = 12t$

$v = 6t^2 + C$

when $t=0, v=-8$

$-8 = C$

$v = 6t^2 - 8$

$x = 2t^3 - 8t + C$

when $t = 0$, $x = 0$

$C = 0$

$\therefore x = 2t^3 - 8t$ m.

ii) when $x = 0$

$0 = 2t^3 - 8t$

$0 = 2t(t^2 - 4)$

$0 = 2t(t-2)(t+2)$

$t = -2, 0, 2$

particle is next at the origin after 2 seconds.

c) i) $\frac{d}{dx}\left(\frac{3}{x^3+1}\right)$

$= \frac{d}{dx}\left(3(x^3+1)^{-1}\right)$

$= -3(x^3+1)^{-2} \cdot 3x^2$

$= \frac{-9x^2}{(x^3+1)^2}$

ii) $V = \pi \int_a^b y^2 \, dx$

$V = \pi \int_0^1 \left(\frac{3x}{x^3+1}\right)^2 dx$

$= \pi \int_0^1 \frac{9x^2}{(x^3+1)^2} \, dx$

$= -\pi \int_0^1 \frac{-9x^2}{(x^3+1)^2} \, dx$

$= -\pi \left[\frac{3}{x^3+1}\right]_0^1$

$= -\pi \left[\frac{3}{(1)^3+1} - \frac{3}{(0)^3+1}\right]$

$= \frac{3\pi}{2}$ units^3

d) i) $V = \pi r^2 h$

$\pi r^2 h = 800$

$h = \dfrac{800}{\pi r^2}$ ——— (1)

$A = 2\pi r^2 + 2\pi r h$ ——— (2)

sub (1) into (2)

$A = 2\pi r^2 + 2\pi r \left(\dfrac{800}{\pi r^2}\right)$

$A = 2\pi r^2 + \dfrac{1600}{r}$

ii) $A = 2\pi r^2 + 1600 r^{-1}$

$\dfrac{dA}{dr} = 4\pi r - 1600 r^{-2}$

$\dfrac{d^2A}{dr^2} = 4\pi + 3200 r^{-3}$

For stat. points let $\dfrac{dA}{dr} = 0$

$4\pi r - \dfrac{1600}{r^2} = 0$

$4\pi r^3 - 1600 = 0$

$4\pi r^3 = 1600$

$r^3 = \dfrac{400}{\pi}$

$r = \sqrt[3]{\dfrac{400}{\pi}}$

$\dfrac{d^2A}{dr^2} = 4\pi + \dfrac{3200}{r^3}$

clearly $\dfrac{d^2A}{dr^2} > 0$ when $r > 0$

$\therefore$ Minimum A when $r = \sqrt[3]{\dfrac{400}{\pi}}$

Minimum $A = 2\pi\left(\sqrt[3]{\dfrac{400}{\pi}}\right)^2 + \dfrac{1600}{\left(\sqrt[3]{\dfrac{400}{\pi}}\right)}$

Minimum Cost $= \left(2\pi\left(\sqrt[3]{\dfrac{400}{\pi}}\right)^2 + \dfrac{1600}{\left(\sqrt[3]{\dfrac{400}{\pi}}\right)}\right) \times 78$

$= \$37211$ (to nearest dollar)